

CHAPTER 4 WEB APPENDIX

Calculus Derivation of Perpetuity Formula

The present value of a perpetuity is given by:

$$PV = \frac{C}{(1+r)} + \frac{C}{(1+r)^2} + \frac{C}{(1+r)^3} + \dots \quad (4A.1)$$

Now multiply both sides of this equation by $(1+r)$ to get:

$$\begin{aligned} PV(1+r) &= \frac{C}{(1+r)}(1+r) + \frac{C}{(1+r)^2}(1+r) + \frac{C}{(1+r)^3}(1+r) + \dots \\ &= C + \frac{C}{(1+r)} + \frac{C}{(1+r)^2} + \frac{C}{(1+r)^3} + \dots \end{aligned} \quad (4A.2)$$

Next subtract (4A.1) from (4A.2)

$$\begin{aligned} PV(1+r) - PV &= C + \frac{C}{(1+r)} + \frac{C}{(1+r)^2} + \frac{C}{(1+r)^3} + \dots \\ &\quad - \frac{C}{(1+r)} - \frac{C}{(1+r)^2} - \frac{C}{(1+r)^3} - \dots \\ &= C \end{aligned} \quad (4A.3)$$

Simplifying provides our result:

$$\begin{aligned} PV(1+r) - PV &= C \\ \Rightarrow PV &= \frac{C}{r} \end{aligned}$$

Growing Perpetuity

In general, if the interest rate promised by the bank is r , and you were to put $\frac{C}{r-g}$ into the bank and initially withdrew C at the end of the first period and then let your withdrawals grow at a rate of g per annum, you could do this forever. The reason is that the principal amount would grow at exactly the rate g , thereby financing the future growth in withdrawals. To see this explicitly, consider what happens at the end of the first period:

$$\frac{C}{r-g}(1+r) - C = \frac{C(1+r) - C(r-g)}{r-g} = \frac{C}{r-g}(1+g)$$

At the end of the second period the principal remaining will be:

$$\frac{C(1+g)}{r-g}(1+r) - C(1+g) = \frac{C(1+g)[(1+r) - C(r-g)]}{r-g} = \frac{C(1+g)^2}{r-g}$$

At the end of the third period the principal remaining will be:

$$\frac{C(1+g)^2}{r-g}(1+r) - C(1+g)^2 = \frac{C(1+g)^2[(1+r) - (r-g)]}{r-g} = \frac{C(1+g)^3}{r-g}$$

notation	
PV	present value; annuity spreadsheet notation for the initial amount
C	cash flow
r	interest rate
g	growth rate
N	date of the last cash flow in a stream of cash flows

Clearly, at the end of N periods the remaining principal would have grown to

$$\frac{C(1+g)^N}{r-g}$$

which is exactly the correct growth rate to finance the required withdrawal of $C(1+g)^N$ at the end of the next period ($N+1$):

$$\frac{C(1+g)^N}{r-g}(1+r) - C(1+g)^N = \frac{C(1+g)^N[(1+r) - (r-g)]}{r-g} = \frac{C(1+g)^{N+1}}{r-g}$$

So the law of one price demands that if the interest rate is r , a growing perpetuity that pays C , growing at rate $g < r$ forever, must have a present value of

$$PV = \frac{C}{(1+r)} + \frac{C(1+g)}{(1+r)^2} + \frac{C(1+g)^2}{(1+r)^3} + \dots = \frac{C}{r-g} \quad (4A.4)$$

Another Derivation of the Growing Perpetuity Formula

The growing perpetuity formula can also be derived by writing a growing perpetuity as a regular perpetuity and then using the formula for the present value of a perpetuity. We first need the following definition:

$$\frac{1}{1+R} \equiv \frac{1+g}{1+r} \quad \text{so } R = \frac{1+r}{1+g} - 1 \quad (4A.5)$$

The next step is to substitute this expression into the formula for the present value of a growing stream of cash flows:

$$\begin{aligned} PV &= \sum_{n=1}^{\infty} \frac{C(1+g)^{n-1}}{(1+r)^n} \\ &= \frac{C}{1+r} + \sum_{n=2}^{\infty} \frac{C(1+g)^{n-1}}{(1+r)^n} \\ &= \frac{C}{1+r} + \frac{1}{1+r} \sum_{n=1}^{\infty} \frac{C(1+g)^n}{(1+r)^n} \\ &= \frac{C}{1+r} + \frac{1}{1+r} \sum_{n=1}^{\infty} \frac{C}{(1+R)^n} \end{aligned} \quad (4A.6)$$

Now the summation in the second term is the present value of a regular perpetuity with interest rate R and payment C . Applying the formula for present value of a regular perpetuity we get:

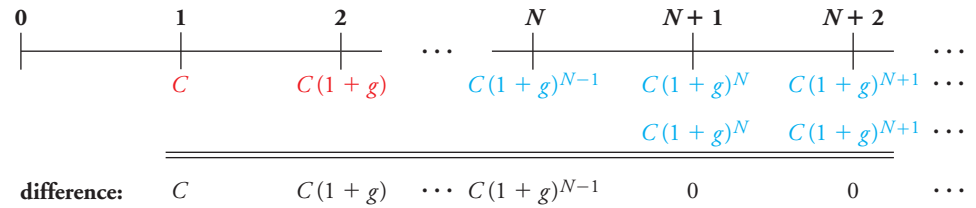
$$PV = \frac{C}{1+r} + \frac{1}{1+r} \sum_{n=1}^{\infty} \frac{C}{(1+R)^n} = \frac{C}{1+r} + \frac{1}{1+r} \left(\frac{C}{R} \right) \quad (4A.7)$$

Now we substitute for R using (4A.5) and simplify:

$$\begin{aligned}
 PV &= \frac{C}{1+r} + \frac{1}{1+r} \left(\frac{C}{\frac{1+r}{1+g} - 1} \right) \\
 &= \frac{C}{1+r} \left(1 + \frac{1+g}{r-g} \right) \\
 &= \frac{C}{1+r} \left(\frac{r-g+1+g}{r-g} \right) \\
 &= \frac{C}{r-g}
 \end{aligned} \tag{4A.8}$$

Present Value of a Growing Annuity

As before, we will create the growing annuity out of two growing perpetuities. The first growing perpetuity (in red on the timeline) begins today (which means that the first payment occurs in period 1) and the second perpetuity (in blue on the timeline) begins in period N (which means the first payment on this perpetuity occurs in period $N+1$). Both these perpetuities are shown on the following timeline:



Both perpetuities grow at rate g , but note that the first cash flow of the second perpetuity is $C(1+g)^N$. The timeline also shows the difference between the cash flows of these two perpetuities—it is precisely the N -period growing annuity we are trying to value. By the Law of One Price, the present value of an N -period growing annuity must be the difference between the present values of the two growing perpetuities. So to get the value of the annuity we simply subtract the present value of the two perpetuities. The present value of the first growing perpetuity (the one that begins today) is $\frac{C}{r-g}$.

To value the second perpetuity we first pretend that we are currently in period N . Recall that the first payment is $C(1+g)^N$, so applying the formula we get the present value in period N to be $\frac{C(1+g)^N}{r-g}$. We then use the second rule of time travel to find the value today. That is, discounting N -periods we get:

$$\frac{C(1+g)^N}{(1+r)^N} = \frac{C}{r-g} \left(\frac{1+g}{1+r} \right)^N \tag{4A.9}$$

By the Law of One Price, the present value of the annuity is given by the difference in the values of these two perpetuities:

$$\begin{aligned} PV &= \text{PV of a growing perpetuity that begins today} \\ &\quad - \text{PV of a growing perpetuity that begins in period } N \\ &= \frac{C}{r-g} - \frac{C}{r-g} \left(\frac{1+g}{1+r} \right)^N \\ &= \frac{C}{r-g} \left(1 - \left(\frac{1+g}{1+r} \right)^N \right) \end{aligned} \tag{4A.10}$$